

## GLOBAL JOURNAL OF ENGINEERING SCIENCE AND RESEARCHES BIHARMONIC PROBLEMS AND THEIR APPLICATIONS

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### ABSTRACT

In the present paper we study some properties of solutions of biharmonic problems. Namely, we study the boundary value problem Navier and a boundary problem for the biharmonic equation. For solving these biharmonic problems with application, in particular, to radar imaging, we need to solve boundary value problems Dirichlet and Neumann for the Poisson equation using the scattering model. In order to select suitable solutions, we solve the Poisson equation with the corresponding boundary conditions Dirichlet and Neumann, that is, some criterion function is minimized in the Sobolev norms. Under appropriate smoothness assumptions, these problems may be reformulated as boundary value problems for the biharmonic equation.

**Keywords:** Biharmonic operator, boundary value problems, scattering model, variational methods.

**MSC:** 35J15; 35J35; 35J40; 58J50; 58J90.

### 1. INTRODUCTION

Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$ , be a bounded Lipschitz domain with connected boundary  $\partial\Omega$ , and  $\Omega \cup \partial\Omega = \bar{\Omega}$  is the closure of  $\Omega$ . We consider the following boundary value problems for the biharmonic equation in Lipschitz domains:

$$\Delta^2 u = f, \quad x \in \Omega \quad (1)$$

with the Navier boundary conditions

$$\begin{cases} u = g_1 & \text{on } \partial\Omega, \\ \sigma \Delta u + (1 - \sigma) \frac{\partial^2 u}{\partial \nu^2} = g_2 & \text{on } \partial\Omega, \end{cases} \quad (2)$$

or with the following (R. Farwig) boundary conditions

$$\begin{cases} \frac{\partial u}{\partial \nu} = h_1 & \text{on } \partial\Omega, \\ \frac{\partial \Delta u}{\partial \nu} = h_2 & \text{on } \partial\Omega, \end{cases} \quad (3)$$

where  $\nu$  is the outer unit normal vector to the domain with the Lipschitz boundary  $\partial\Omega$ . The coefficient  $\sigma$  is a constant known as the Poisson ratio,  $\frac{1}{n-1} < \sigma < 1$ .

For  $n = 2$ , these problems and also the Neumann problem are related to the study of the transverse vibrations of a thin plate with a free edge and which occupies at rest a planar region of shape  $\partial\Omega$ . The coefficient  $\sigma$  represents the Poisson's ratio of the material that the plate is made of. For more details on the physical interpretation of the Neumann problem and on the Poisson's ratio  $\sigma$ , we refer, for example, to [3]. Note the paper [4], where the author studies the dependence of the vibrational modes of a plate subject to homogeneous boundary conditions upon the Poisson's ratio  $0 < \sigma < \frac{1}{2}$ , providing also a perturbation formula for the frequencies as functions of the Poisson's coefficient.

Note that standard elliptic regularity results are available in [7]. This monograph covers higher order linear and nonlinear elliptic boundary value problems, mainly with the biharmonic (polyharmonic) operator as leading principal part. Underlying models and, in particular, the role of different boundary conditions are explained in detail. As for linear problems, after a brief summary of the existence theory and  $L^p$  and Schauder estimates, the focus is on positivity. The required kernel estimates are also presented in detail.

In [6] and [7], the spectral and positivity preserving properties for the inverse of the biharmonic operator under Steklov and Steklov–type boundary conditions are studied. These are connected with the first Steklov eigenvalue. It is shown that the positivity preserving property is quite sensitive to the parameter involved in the boundary condition.

Elliptic problems with parameters in the boundary conditions are called Steklov problems from their first appearance in [30]. In the case of the biharmonic operator, these conditions were first considered in [2], [11], who studied the isoperimetric properties of the first eigenvalue.

In [5], the boundary value problems for the biharmonic equation and the Stokes system are studied in a half space, and, using the Schwartz reflection principle in weighted  $L^q$ -space the uniqueness of solutions of the Stokes system or the biharmonic equation is proved.

Boundary value problems for a biharmonic (polyharmonic) equation and for the elasticity system in unbounded domains are studied in [12]–[21], in which the condition of the boundedness of the following weighted Dirichlet integral of solution is finite, namely

$$\int_{\Omega} |x|^a |\partial^\alpha u|^2 dx < \infty, \quad a \in \mathbb{R},$$

where  $a \in \mathbb{R}$  is a fixed number and  $|\partial^\alpha u|^2$  denotes the Frobenius norm of the Hessian matrix of  $u$ . In particular, in [12]–[21] has been studied the dimension of the space of the solutions to the boundary value problems for a biharmonic (polyharmonic) equation and for the elasticity system, providing explicit formulas which depends on  $n$  and  $a$ .

In [8]–[10], generalizations of the Hardy inequality were established for bounded and for a wide class of unbounded domains, and applied these to investigate boundary value problems for elliptic equations and systems. In particular, the problems of the existence, the uniqueness, the stability and the asymptotic expansions of solutions of boundary value problems were studied.

**Notation:**  $C_0^\infty(\Omega)$  is the space of infinitely differentiable functions in  $\Omega$  with compact support in  $\Omega$ ;  $H^m(\Omega)$  is the Sobolev space obtained by the completion of  $C^\infty(\overline{\Omega})$  with respect to the norm

$$\|u; H^m(\Omega)\| = \left( \int_{\Omega} \sum_{|\alpha| \leq m} |\partial^\alpha u|^2 dx \right)^{1/2}, \quad m = 1, 2,$$

where  $\partial^\alpha \equiv \partial^{|\alpha|} / \partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$ ,  $\alpha = (\alpha_1, \dots, \alpha_n)$  is a multi-index,  $\alpha_i \geq 0$  are integers, and  $|\alpha| = \alpha_1 + \dots + \alpha_n$ ;  $H^m(\Omega)$  is the space obtained by the completion of  $C_0^\infty(\Omega)$  with respect to

the norm  $\|u; H^m(\Omega)\|; \check{H}_{loc}^m(\Omega)$  is the space obtained by the completion of  $C_0^\infty(\Omega)$  with respect to the family of semi-norms

$$\|u; H^m(\Omega \cap B_0(R))\| = \left( \int_{\Omega \cap B_0(R)} \sum_{|\alpha| \leq m} |\partial^\alpha u|^2 dx \right)^{1/2}$$

for all open balls  $B_0(R) := \{x : |x| < R\}$  in  $\mathbb{R}^n$  for which  $\Omega \cap B_0(R) \neq \emptyset$ . Finally  $H^{1/2}(\partial\Omega)$  is the usual trace space on the boundary and  $H^{-1/2}(\partial\Omega)$  is its dual (see, for ex., [1]).

If we set  $\sigma = 1$ , for the biharmonic equation the problems reads, with the Navier boundary conditions

$$\begin{cases} u = g_1 & \text{on } \partial\Omega, \\ \Delta u = g_2 & \text{on } \partial\Omega, \end{cases} \quad (4)$$

or with the following (R. Farwig) boundary conditions

$$\begin{cases} \frac{\partial u}{\partial \nu} = h_1 & \text{on } \partial\Omega, \\ \frac{\partial \Delta u}{\partial \nu} = h_2 & \text{on } \partial\Omega. \end{cases} \quad (5)$$

**Definition 1.** A solution of the biharmonic equation (1) in  $\Omega$  is a function  $u \in H^2(\Omega)$  such that, for every function  $\varphi \in C_0^\infty(\Omega)$ , the following integral identity holds:

$$\int_{\Omega} \Delta u \Delta \varphi dx = \int_{\Omega} f \varphi dx, \quad f \in L^2(\Omega). \quad (6)$$

**Definition 2.** A function  $u$  is a solution of the Navier problem (1),(4) with  $g_1 = g_2 = 0$ , if  $u \in H^2(\Omega) \cap \overset{\circ}{H}^1(\Omega)$  such that the integral identity (6) holds for every function  $\varphi \in C_0^\infty(\Omega)$ .

**Definition 3.** A function  $u$  is a solution of the Farwig problem (1),(5) with  $h_1 = h_2 = 0$ , if  $u \in \overset{\circ}{H}^2(\Omega)$ ,  $\partial u / \partial \nu = 0$  on  $\partial\Omega$ , and the integral identity (6) holds for every function  $\varphi \in \overset{\circ}{H}^2(\mathbb{R}^n)$  such that  $\partial \varphi / \partial \nu = 0$  on  $\partial\Omega$ .

## 2. SCATERING MODEL

In the section we derive the mathematical model used for describing the radar process. In our parametrization the unknown is the height function  $H$ . As will be shown the height function is determined in two steps. In the first step  $\mathfrak{L}(H)$ , with  $\mathfrak{L}$  a certain second-order differential operator, is determined. After retrieving  $H$  the equation  $\mathfrak{L}(H) = f$  must be solved. To a good approximation the operator  $\mathfrak{L}$  can be replaced by the Laplacian. So the second step simply consists of solving the Poisson equation over some smooth bounded domain, usually a rectangular region in the plane. The problem here is that no natural boundary conditions are available.

Here we will briefly discuss the mathematical inverse problem to be resolved in order to recover the ground topography height function from radar data. First cylindrical coordinates  $(r, \varphi, z)$  are introduced according to Fig. 1, where it is understood that the aircraft is flying at a constant speed along the  $z$ -axis. Further  $r$  denotes the distance from a point on the ground surface to the  $z$ -axis and  $\varphi$  is the angle between radius vector and a horizontal plane through the  $z$ -axis. Then the ground surface may be described by a function  $H(r, z)$  through the equation

$$\frac{H(r, z)}{r} - \varphi = 0. \quad (7)$$

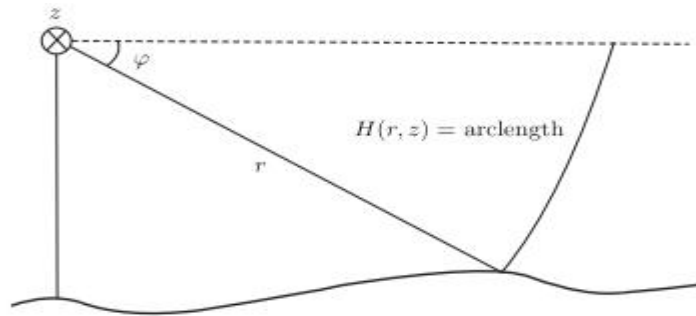


Figure 1. The ground surface measured at a fixed aircraft position.

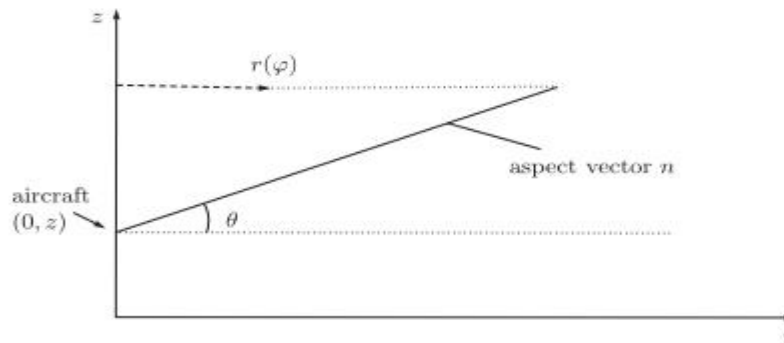


Figure 2. The measuring geometry as seen from above.

When  $r$  is large,  $-H(r, z)$  is approximately a Cartesian height function. Fig. 2 shows a top view of the same scene. We have also indicated an aspect vector from the aircraft to some point on the ground, forming an angle  $\theta$  with a vertical plane through the aircraft. Normalized to unit length, the aspect vector is denoted by  $\hat{n}$ . Accordingly

$$\hat{n} = \cos \theta \hat{r}(\varphi) + \sin \theta \hat{z}. \tag{8}$$

Here  $\hat{r}(\varphi)$  denotes the cylindrical unit basis vector corresponding to the  $r$ -coordinate for the ground point as shown in the Fig. 2. For a point on the ground surface with coordinates  $(r, \varphi, z)$  we obtain, from Eq. (7), the following expression for the ground surface normal  $\bar{m}$ ,

$$\bar{m} = \text{grad} \left( \frac{H(r, z)}{r} - \varphi \right) = \frac{\partial(H/r)}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial H}{\partial z} \hat{z} - \frac{1}{r} \hat{\varphi}. \tag{9}$$

Let  $\hat{m}$  denote the normalized normal. Then

$$\hat{m} \circ \hat{n} = \left( r \cos \theta \frac{\partial(H/r)}{\partial r} + \sin \theta \frac{\partial H}{\partial z} \right) : \sqrt{1 + \left( \frac{\partial(H/r)}{\partial r} \right)^2 + \left( \frac{\partial H}{\partial z} \right)^2}. \tag{10}$$

Note that  $(r, \varphi, z)$  in Eq. (10) are related to the ground surface point and not to the position of the aircraft.

Let  $(z_0, 0)$  be a position of the aircraft and  $R$  the distance to some point on the surface. According to Fig. 3 the coordinates  $(r, z)$  are then equal to  $(z_0 + R \sin \theta, R \cos \theta)$ . Next, to

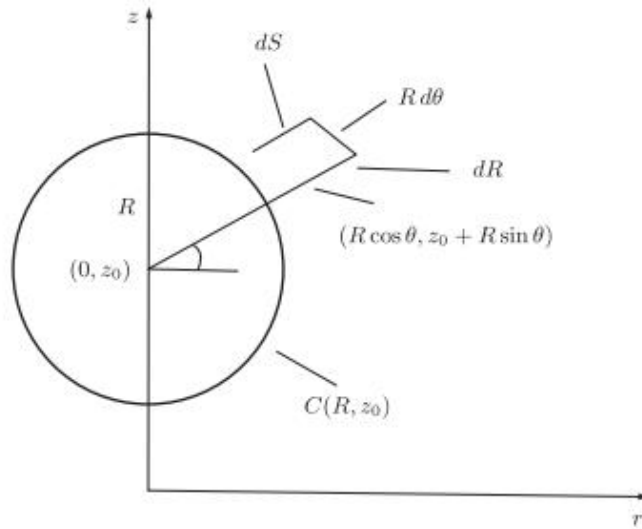


Figure 3. The coordinate system used to describe an infinitesimal surface element,  $dS$ .

obtain a scattering model we will assume that the reflectivity from a ground surface element (see Fig. 4) is

$$\approx \frac{\hat{m} \circ \hat{n}}{R} dR d\theta. \tag{11}$$

From Fig. 4, where a vertical plane through  $(z_0, 0)$  (the aircraft) and the ground point  $(z_0 + R \sin \theta, R \cos \theta)$  is displayed, we conclude that the solid angle  $d\Omega$  under which the surface element  $dS$  is seen from the antenna is approximately

$$\frac{dR \cos \alpha R d \theta}{R^2} = - \frac{\hat{m} \circ \hat{n}}{R} dR d\theta.$$

In expression (11) we are consequently assuming that the local reflectivity is proportional to the solid angle occupied by the infinitesimal surface element  $dS$ . The total reflected signal  $G(R, z_0)$  from all points at a distance  $R$  from the antenna may now be obtained by integration over the circle  $C(R, z_0) = \{(r, z) : (z - z_0)^2 + r^2 = R^2\}$  in Fig. 3.

$$G(R, z_0)dR = c \int_{-\pi}^{\pi} \frac{\hat{m} \circ \hat{n}(z_0 + R \cos \theta, R \sin \theta)}{R} d\theta dR$$

i.e.

$$RG(R, z_0) = c \int_{-\pi}^{\pi} \hat{m} \circ \hat{n}(z_0 + R \cos \theta, R \sin \theta) d\theta. \tag{12}$$

Assuming that  $\hat{m} \circ \hat{n}$  is small Eq. (10) may be replaced by

$$\hat{m} \circ \hat{n} = r \cos \theta \frac{\partial(H/r)}{\partial r} + \sin \theta \frac{\partial H}{\partial z}.$$

By inserting this into Eq. (12) we get, after multiplying by  $R$ ,

$$R^2G(R, z_0) = c \int_{-\pi}^{\pi} (rR \cos \theta \frac{\partial(H/r)}{\partial r} + R \sin \theta \frac{\partial H}{\partial z}) d\theta.$$

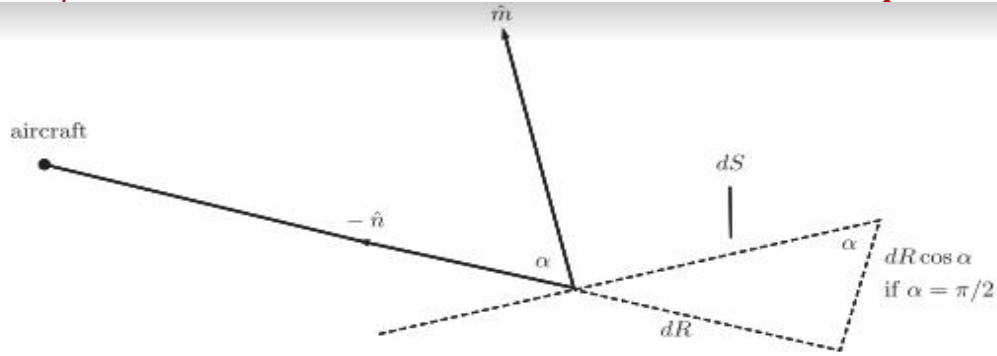


Figure 4. The infinitesimal surface element,  $dS$ , as it is seen from the aircraft.

Using the parametrization

$$z = z_0 + R \sin \theta, \quad r = R \cos \theta,$$

this may be rewritten as a curve integral over  $C(R, z_0)$ , with  $dz = R \cos \theta d\theta$  and  $dr = -R \sin \theta d\theta$ ,

$$R^2 G(R, z_0) = c \int_{C(R, z_0)} \left( r \frac{\partial(H/r)}{\partial r} dz - \frac{\partial H}{\partial z} dr \right). \tag{13}$$

By applying Green's formula we get

$$R^2 G(R, z_0) = c \iint_{D(R, z_0)} \mathfrak{L}(H)(r, z) dz dr, \tag{14}$$

where  $D$  is the disc,

$$D(R, z_0) = \{(r, z) : (z - z_0)^2 + r^2 \leq R^2\}$$

and

$$\mathfrak{L}(H) = \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} (H/r) \right) + \frac{\partial^2}{\partial z^2} (H). \tag{15}$$

The problem of finding the height function  $H$  from radar data  $G(r, z)$  may now be divided into two parts:

- (a) First solve the integral equation (14) for  $\mathfrak{L}(H)(r, z) = f(r, z)$ .
- (b) Next solve the partial differential equation

$$\mathfrak{L}(H) = f \tag{16}$$

for  $H$ . We note that if  $r$  is large and if  $\hat{m} \circ \hat{n}$  is small it is reasonable to make the approximation

$$\mathfrak{L}(H) \approx \frac{\partial^2 H}{\partial r^2} + \frac{\partial^2 H}{\partial z^2} = \Delta H$$

so that Eq. (16) becomes Poisson's equation. To consider the first problem (a), both members in Eq. (14) are differentiated with respect to  $R$ . Then we get

$$\frac{1}{R} \frac{d}{dR} (R^2 G(R, z_0)) = c \int_{-\pi}^{\pi} \mathfrak{L}(H)(z_0 + R \cos \nu, R \sin \nu) d\nu,$$

where the right-hand side is proportional to the average of  $\mathfrak{L}(H)$  over the circle  $C(R, z_0)$ . In [2] an explicit solution is given for this problem of recovering the function  $\mathfrak{L}(H)(r, z)$  when the average of  $\mathfrak{L}(H)$  is known for all circles  $C(R, z_0)$  with center on the  $z$ -axis and with arbitrary radius  $R$ . The solution formula is

$$\mathfrak{L}(H)^{(F,F)}(\sigma, \omega) \sim |\omega| \left[ \frac{1}{R} \frac{d}{dR} \{R^2 G(r, z)\} \right]^{(F,H_0)}(\sigma, \sqrt{\omega^2 + \sigma^2}). \tag{17}$$

Here the notation  $(F, F)$  means that we have taken the Fourier transform with respect to both the variables and  $(F, H_0)$  means that we have taken Fourier transform with respect to the first variable and the Hankel-zero transform with respect to the second. After some calculations Eq. (17) may be rewritten

$$\mathfrak{L}(H)^{(F,F)}(\sigma, \omega) \sim |\omega| \sqrt{\omega^2 + \sigma^2} [RG(r, z)]^{(F,H_1)}(\sigma, \sqrt{\omega^2 + \sigma^2}). \tag{18}$$

Formula (18) may now be used in order to recover the function  $\mathfrak{L}(H)$  in spatial coordinates. Of course, approximating  $\mathfrak{L}(H)$  by  $\Delta H$  we could rewrite Eq. (18) as

$$H^{(F,F)}(\sigma, \omega) \sim |\omega| \frac{1}{\sqrt{\omega^2 + \sigma^2}} [RG(r, z)]^{(F,H_1)}(\sigma, \sqrt{\omega^2 + \sigma^2}), \tag{19}$$

where  $H_1$  denotes that we have taken the Hankel-one transform with respect to the second variable. Then we could obtain  $H$  directly by a two dimensional Fourier transform. However, our solution might be expected to have errors caused by, e.g. noisy radar data and errors caused by the particular numerical implementation of the inversion formula (17) (or Eq. (18)) and therefore we would rather prefer to divide the solution procedure into the two steps described above and to use the second step, the solution of Poisson’s equation, so that we perform some kind of regularization of the final solution. Note also that by using Eq.(19) as our solution formula we have tacitly assumed periodic boundary conditions for the Poisson equation.

### 3. SOLUTION CONCEPTS FOR THE POISSON EQUATION

In this section we discuss different possibilities of defining a unique height function. Essentially our approach consists in minimizing some norm of the solution provided that it also satisfies the Poisson equation. In particular we consider the  $L^2$ - and  $H^1$ -norms. We also show how these two optimization problems may be reformulated as boundary value problems for the biharmonic equation. Note that the corresponding Poisson problem is well-posed unless  $\sigma = 1$ .

In the domain  $\Omega$  for the Poisson equation we consider the following boundary value problems

$$\Delta u = f, \quad x \in \Omega \tag{20}$$

with the Dirichlet boundary condition

$$u = g \quad \text{on} \quad \partial\Omega, \tag{21}$$

or the Neumann boundary conditions

$$\nabla u \cdot \nu = h \quad \text{on} \quad \partial\Omega, \tag{22}$$

where  $\nu$  is the outer unit normal vector to  $\partial\Omega$ .

The boundary operators are independent of any particular choice of orientation for the rectangular coordinate systems. Finally, for  $\Omega$  a rectangular region in, e.g., the plane

$$\Omega = \{(x, y) : a < x < b, c < y < d\}, \tag{23}$$

there may be the following boundary conditions

$$u(a, y) = u(b, y), \quad u(x, c) = u(x, d), \tag{24}$$

or the periodic boundary conditions

$$u_x(a, y) = u_x(b, y), \quad u_y(x, c) = u_y(x, d). \tag{25}$$

Provided  $g$  is smooth enough boundary conditions (21) define a unique solution of Eq. (20). For (22) and (25) the solution is determined up to a constant. It is also possible to use different mixtures of these three types of boundary conditions. Note that for cases (22) and (25) the following consistency conditions must hold, respectively:

$$\int_{\Omega} f \, dx = \int_{\partial\Omega} h \, ds \quad \text{and} \quad \int_{\Omega} f \, dx = 0.$$

We now consider a different way to select a solution to Eq. (20). Here we use a criterion function and optimize this criterion over the set of solutions to the Poisson equation. Scattering model of Section 2 shows the physical interpretation of function  $u(x, y)$  is a surface function. We need to pick out the smoothest surface (in some sense) that fulfills Eq. (20), using the Sobolev space norms as criterion functions. Denote by  $V_{f,i}$  the following set:

$$V_{f,i} = \{u \in H^i(\Omega) : \Delta u = f, f \in L^2(\Omega)\}, \quad i = 0, 1, 2, \tag{26}$$

where  $H^0(\Omega) = L^2(\Omega)$ . The equality  $\Delta u = f$  is to be interpreted in the sense of distributions, i.e.,

**Definition 4.** A solution of the Poisson equation (20) in  $\Omega$  is a function  $u \in H^1(\Omega)$  such that the following integral identity holds:

$$\int_{\Omega} u \Delta \varphi \, dx = \int_{\Omega} f \varphi \, dx, \quad \forall \varphi \in C_0^\infty(\Omega).$$

**Lemma 1.**  $V_{f,i}$  is a closed, convex and nonempty set of  $H^i(\Omega)$ .

Let  $\alpha$  be a multi-index and  $\beta_1 > 0$  a given parameter. We consider the following optimization problems:

$$I_0(u) \equiv \min_{u \in V_{f,0}} \int_{\Omega} |u|^2 \, dx, \tag{27}$$

and

$$I_1(u) \equiv \min_{u \in V_{f,1}} \int_{\Omega} |u|^2 \, dx + \beta_1 \int_{\Omega} \sum_{|\alpha|=1} |\partial^\alpha u|^2 \, dx. \tag{28}$$

**Theorem 1.** Problems (27) and (28) have unique solutions  $u_0$  and  $u_1$ , respectively.

From problems (27) and (28) we have the following results characterizing the solutions.

**Theorem 2.** Let  $u_0 = \Delta v$ . For the solution  $u_0$  of the problems (27), where  $v \in H^2(\Omega) \cap \overset{\circ}{H}^1(\Omega)$  is the unique solution of the following biharmonic problem

$$\begin{cases} \Delta^2 v = f & \text{on } \Omega, \\ v = \Delta v = 0 & \text{on } \partial\Omega. \end{cases} \tag{29}$$



**Theorem 3.** Let  $u_1 = \Delta v$ . For the solution  $u_1$  of the problems (28), where  $v \in H^2(\Omega)$  is the unique solution in the class  $\{\psi \in H^1(\Omega) : \Delta\psi \in H^1(\Omega)\}$  of the following biharmonic problem

$$\begin{cases} \Delta^2 v = f & \text{in } \Omega, \\ v = \beta_1 \Delta v, \nabla v \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases} \quad (30)$$

We conclude this section by a theorem relating the solution of problems (27) and (28). First we recall the following definition.

**Definition 5.**  $\Omega \subset \mathbb{R}^n$  is called star-shaped if there exists  $x_0 \in \Omega$  such that for all  $x \in \Omega$  the set  $\{t \in \mathbb{R} : x_0 + t(x - x_0) \in \Omega\}$  is an interval.

**Theorem 4.** Assume that  $\Omega \subset \mathbb{R}^n$  is open, bounded and star-shaped. If  $u_{1,\beta_1} \in H^1(\Omega)$  denotes the solution of problem (28) with the parameter  $\beta_1 > 0$ , and if  $u_0 \in L^2(\Omega)$  denotes the solution of problem (27), then

$$u_{1,\beta_1} \rightarrow u_0 \quad \text{in } L^2(\Omega) \quad \text{as } \beta_1 \rightarrow 0+.$$

The proof of Theorems 1–4 is carried out in the same way as it was done for other boundary value problems in the article of the authors [27].

#### 4. APPLICATIONS

As applications, in [23], [26], the eigenvalue problems of the symmetric tensor-block matrix of any even rank and sizes  $2 \times 2$  is studied. Some definitions and theorems are formulated concerning the tensor-block matrix. Formulas expressing the classical invariants of the tensor-block matrix of any even rank and sizes  $2 \times 2$  through the first invariants of the powers of this tensor-block matrix and the canonical representation of the tensor-block matrix are given. Besides, a classification of the micro-polar linear elastic anisotropic bodies that do not have a center of symmetry is given. In [24], some questions about the parametrization of three-dimensional thin body with one small size under an arbitrary base surface and the changing of transverse coordinate from  $-1$  to  $1$  are considered. The vector parametric equation of the thin body domain is given. Note the paper [25], where the author considers questions on the decomposition of the initial boundary value problems of elasticity theory and thin bodies for some anisotropic media, and from three-dimensional decomposed initial boundary value problems the corresponding decomposed initial boundary value problems for the theory of thin bodies are obtained. In [22] the expression is presented for the propagation velocities of waves of a certain medium through the eigenvalues of the material tensors, as well as expressions for the propagation velocities of waves for a micro-polar medium with anisotropy and isotropic micro-polar material are obtained.

#### 5. ACKNOWLEDGMENTS

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